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A weakly homogeneous rigid space

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Abstract

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A space is weakly homogeneous if every pair of points have homeomorphic neighborhoods. A space is rigid if it has no self-homeomorphisms. We construct a weakly homogeneous rigid space. Unlike previously known examples, ours is constructed in ZF.

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1. Introduction

A space which is homogeneous and rigid has only one point. A space X is homogeneous if given points $p_1, p_2 \in X$, there is a homeomorphism $f: X \to X$ such that $f(p_1) = p_2$. X is merely weakly homogeneous if given points p_1, p_2 there are neighborhoods $U_1, U_2 \subseteq X$, and a homeomorphism $f: U_1 \to U_2$ such that $f(p_1) = p_2$. A space is rigid if its only self-homeomorphism is the identity. In [2], van Mill constructs a remarkable topological group whose only self-homeomorphisms are translations. An open subset of a topological group is always weakly homogeneous, and van Mill notes that removing a point from his group leaves a rigid space, so this gives a nontrivial example of a weakly homogeneous rigid space. The metric space $\mathcal{M}$ that we present below gives another example. $\mathcal{M}$ is noteworthy in that, unlike van Mill's space, its construction can be carried out in ZF. The construction of $\mathcal{M}$ is related to those of Kannan and Rajagopalan [1]. We thank J. van Mill for his comments.

2. The construction

Let W_i be an orientable two-manifold of genus i, $i = 0, 1, 2$, with a compatible metric d_{W_i}. Furthermore, W_0 and W_1 shall have distinguished base points, w_0 and

w_1. Also fix embeddings $g_i: W_i \to C$ of each surface into the open unit cube $C = (0, 1)^3$.

$\{0, 1\}^\omega$ is the set of sequences whose terms are equal to 0 or 1. Define set maps $\phi_i: W_i \to \{0, 1\}^\omega$, $i = 0, 1, 2$ as follows. If $p \in W_i$ and $g_i(p) = (\alpha_1, \alpha_2, \alpha_3)$ and the binary expansion of α_j is $a_{j1}a_{j2}a_{j3}\ldots$, set $\phi_i(p) = a_{11}a_{21}a_{31}a_{12}a_{22}a_{32}a_{13}\ldots$. In cases of ambiguity, the terminating expansion of α_j is taken. The set maps ϕ_i are injective since the embeddings g_i separate points, and this is all we shall need of the ϕ_i; we have defined the ϕ_i as we have to make explicit that such functions may be constructed without the axiom of choice. Sometimes we drop the subscript and write ϕ when the context is clear.

We will describe metric spaces $X_0 \subset X_1 \subset X_2 \subset \cdots$ by induction. Then $\mathcal{M} = \bigcup_{i=0}^{\infty} X_i$, with the metric topology will be the desired space. To facilitate the induction we will associate to every $p \in X_n - X_{n-1}$ an $n+1$-tuple $\gamma(p) = (\beta_0, \ldots, \beta_n)$ where $\beta_k = \beta_{k0}\beta_{k1}\beta_{k2}\ldots \in \{0, 1\}^\omega$. To include the case $n = 0$, we set $X_{-1} = \emptyset$.

X_0 will be W_2, and for $p \in X_0$, set $\gamma(p) = (\phi_2(p))$.

Now suppose we have constructed spaces $X_0 \subset \cdots \subset X_n$, with metric d, and that we have defined $\gamma(p) = (\beta_0, \ldots, \beta_i)$ for each $p \in X_i - X_{i-1}$, $0 \leq i \leq n$.

For each $p \in X_n - X_{n-1}$ we will define a space W^p which will be a copy of W_0 or W_1 and we will denote the distinguished basepoint of W^p by w^p. Then X_{n+1} will be the union of X_n and all the W^p, after identifying each p with w^p. We say that W^p is attached to X_n at p.

Set m to be the exponent of the highest power of 2 dividing n. If $\gamma(p) = (\beta_0, \ldots, \beta_n)$, W^p will be a copy of W_i, $i = \beta_{m, \frac{1}{2}(n \cdot 2^{-m} - 1)}$. The purpose of this is made clear below.

To extend the metric d to X_{n+1} we consider three cases:

Case 1. $q_1 \in X_n$ and $q_2 \in W^p$. Then $d(q_1, q_2) = d(q_1, p) + d_{W^p}(p, q_2)$.

Case 2. $q_1, q_2 \in W^p$. Then $d(q_1, q_2) = d_{W^p}(q_1, q_2)$.

Case 3. $q_1 \in W^{p_1}$ and $q_2 \in W^{p_2}$. Then $d(q_1, q_2) = d_{W^{p_1}}(q_1, p_1) + d(p_1, p_2) + d_{W^{p_2}}(p_2, q_2)$.

Finally, if $q \in X_{n+1} - X_n$ and $q \in W^p$ and $\gamma(p) = (\beta_0, \ldots, \beta_n)$, set $\gamma(q) = (\beta_0, \ldots, \beta_n, \phi(q))$. This completes the induction and the construction of $\mathcal{M}$.

3. Rigidity

Lemma 1. *If* $\gamma(q_1) = \gamma(q_2)$, *then* $q_1 = q_2$.

Proof. By induction. If $q_1, q_2 \in X_0$, then $\phi_2(q_1) = \phi_2(q_2)$ and $q_1 = q_2$ by the injectivity of ϕ_2.

If $\gamma(q_1) = \gamma(q_2)$, then they must both be $n+1$-tuples for some n and $q_1, q_2 \in X_n - X_{n-1}$. Assume that the lemma is valid for points in X_{n-1}.

Say that $q_1 \in W^{p_1}$, $q_2 \in W^{p_2}$ and $\gamma(p_1) = (\beta_0, \ldots, \beta_n)$, $\gamma(p_2) = (\beta_0', \ldots, \beta_n')$. Then if $\gamma(q_1) = (\beta_0, \ldots, \beta_n, \phi(q_1)) = (\beta_0', \ldots, \beta_n', \phi(q_2)) = \gamma(q_2)$ we must have $\beta_i = \beta_i'$

for $0 \leq i \leq n$, so $p_1 = p_2$ by induction. Also $\phi(q_1) = \phi(q_2)$, so $q_1 = q_2$, by the injectivity of ϕ. $\square$

To show that $\mathcal{M}$ is rigid, we argue that $\gamma(p)$ is a topological invariant of $p \in \mathcal{M}$. Then, by the lemma, a self-homeomorphism must preserve every point, so $\mathcal{M}$ is rigid.

Set $J = \{W^p \mid p \in \mathcal{M}\} \cup \{X_0\}$.

Recall that p is a *cut point* of a connected set V if $V - \{p\}$ is not connected. In this case we say that $\{p, U_1, U_2\}$ is a cutting of V if U_1 and U_2 are disjoint open sets and $V - \{p\} = U_1 \cup U_2$. If $a \in U_1$ and $b \in U_2$ and $\{p, U_1, U_2\}$ is a cutting of V, then we say that p cuts a and b.

Lemma 2. *If Y is a connected subset of $\mathcal{M}$ without cutpoints, then $Y \subseteq Z \in J$, for some Z.*

Proof. Given a connected Y not contained in any $Z \in J$, we will find a cutpoint of Y. Since two distinct elements of J have at most a point in common, there must be $q_1, q_2 \in Y$ so that $\{q_1, q_2\}$ is not contained in any $Z \in J$. Say $q_1 \in X_m - X_{m-1}$ and $q_2 \in X_n - X_{n-1}$. Without loss of generality, say $n \geq m$. Then $n \neq 0$ or else $\{q_1, q_2\} \subset X_0 \in J$. So $q_2 \in W^p$ for some $p \neq q_2$. Of course $p \in W^p$, so $p \neq q_1$ or else $\{q_1, q_2\} \subset W^p \in J$. Then, by the construction p cuts q_1 and q_2. $\square$

By the lemma, a subset of $\mathcal{M}$ which is homeomorphic to W_2 must be X_0, because W_2 is connected, has no cutpoints and is not homeomorphic to any proper subspace of W_2, or any subspace of W_0 or W_1. (If a subspace of a manifold is homeomorphic to a compact space, it must be closed. On the other hand, if it is homeomorphic to a manifold of the same dimension, then it must be an open set, by invariance of domain.) Similarly, a subset of $\mathcal{M}$ homeomorphic to W_0 or W_1 must be one of the W^p. Therefore, the filtration $\mathcal{M} = \bigcup_{i=0}^{\infty} X_i$ is a feature of the topology of $\mathcal{M}$, not just of our construction. The distinguished basepoint of W^p, p, has the property that there is a cutting $\{p, U_1, U_2\}$ of $\mathcal{M}$ with $X_0 \subset U_1$, $W^p - \{p\} \subset U_2$. Since no other point of W^p has this property, the distinguished basepoints may also be recovered from the topology alone.

We use induction once more. Assume that $\gamma(p)$ is determined by the topology of $\mathcal{M}$ for all $p \in X_n$.

Let $q \in X_{n+1}$. Then if $n \neq -1$, $q \in W^p$ for some $p \in X_n$, and $\gamma(p) = (\beta_0, \ldots, \beta_n)$ is determined by the topology of $\mathcal{M}$, by the induction hypothesis. Since $\gamma(q) = (\beta_0, \ldots, \beta_n, \phi(q))$ it remains to see that $\phi(q) \in \{0, 1\}^\omega$ is determined by the topology of $\mathcal{M}$. Say $\phi(q) = a_0 a_1 a_2 \ldots$. We must determine each a_k from the topology of $\mathcal{M}$. Set $l = 2^n(1+2k)$. Let U be the connected component of $\mathcal{M} - \{q\}$ which is disjoint from X_0. Then all the W^p contained in $U \cap X_l$, but not in $U \cap X_{l-1}$ will have genus a_k, by the construction. This determines a_k from the topology of $\mathcal{M}$ for all k, hence also $\phi(q)$ and so also $\gamma(q)$.

We can now conclude that $\mathcal{M}$ is rigid, having associated distinct topological invariants to each point.

4. Weak homogeneity

Let $T_i \subset W_i$, $i=0,1$, be open neigborhoods, homeomorphic to the open unit disc, of the distinguished basepoints of each W_i. If W^p is a copy in M of one of the W_i, T^p will be the subset of W^p corresponding to T_i in W_i. Fix homeomorphisms and $f: T_1 \to T_2$ and $g: T_2 \to T_1$.

Given p_1, $p_2 \in \mathcal{M}$, we must find homeomorphic open neighborhoods of the two points. Say $p_1 \in X_m - X_{m-1}$, $p_2 \in X_n - X_{n-1}$. Let R_0, (S_0) be open neighborhoods, homeomorphic to the open unit disc, of p_1, (p_2) in X_m, (X_n), respectively. Let $h_0: R_0 \to S_0$ be a homeomorphism.

Now we proceed by induction. Assume $h_j: R_j - S_j$ is a homeomorphism between open neighbourhoods R_j, (S_j) of p_1, (p_2) in X_{m+j}, (X_{n+j}), respectively.

Define R_{j+1}, (S_{j+1}) to be the unions of R_j, (S_j) and all the T^p for $p \in R_j - R_{j-1}$, $(p \in S_j - S_{j-1})$, respectively. Clearly R_{j+1}, (S_{j+1}) is an open neighborhood of p_1, (p_2) in X_{m+j+1}, (X_{n+j+1}), respectively.

We define a homeomorphism $h_{j+1}: R_{j+1} \to S_{j+1}$ that extends $h_j: R_j \to S_j$ as follows. If $q \in R_{j+1} - R_j$ and $q \in T^p$, we map q to a point in $T^{h_j(p)}$ using a lifting of either f or g or the identity map on one of the T_i, whichever is available.

Now $R = \bigcup_{j=0}^{\infty} R_j$ and $S = \bigcup_{j=0}^{\infty} S_j$ will be homeomorphic neighborhoods of p_1 and p_2 in $\mathcal{M}$, the homeomorphism $h: R \to S$ being the unique map that extends all the h_j. With a little care we might even take h, and all the h_j, to be isometries.

5. Open problem

It would be very interesting to know whether a weakly homogeneous rigid space could be locally compact, or compact. A connected component of such a space is already such a space, and a weakly homogeneous rigid space which is totally disconnected must be trivial. Known results on continua with many cut points suggest that a compact connected weakly homogenous rigid space would require ideas very different from the above.

References

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